On the shape of vortices for a rotating Bose Einstein condensate

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For a Bose-Einstein condensate placed in a rotating trap, we study the simplified energy of a vortex line derived in [6] in order to determine the shape of the vortex line according to the rotational velocity and the elongation of the condensate. The energy reflects the competition between the length of the vortex which needs to be minimized taking into account the anisotropy of the trap and the rotation term which pushes the vortex along the z axis. We prove that if the condensate has the shape of a pancake, the vortex stays straight along the z axis while in the case of a cigar, the vortex is bent.

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I. INTRODUCTION

Dilute Bose-Einstein condensates have recently been achieved in confined alkali-metal gases and the study of vortices is one of the key issues. One type of experiments consists in imposing a laser beam on the magnetic trap holding the atoms to create a harmonic anisotropic rotating potential [1, 2, 3, 4]. Vortices are nucleated and the number of vortices depends on the rotational velocity. It has been observed experimentally [1], that when the first vortex is nucleated, the contrast is not 100% which means that the vortex line is not straight but bending. Numerical computations of the Gross Pitaevskii equation have shown evidence of vortex bending [5].

The aim of this paper is to characterize the dependence of the shape of the vortex line on the elongation of the trap and the rotational velocity. In particular, using a simplified energy for a vortex line derived in [6] from the Gross Pitaevskii energy, we study the stability and instability of the straight vortex and we prove that when the condensate has a cigar shape the first vortex is bent, while when it is a pancake, the first vortex is straight and lies on the axis of rotation. We also show that vortices cannot be nucleated too close to the boundary, because they have a minimal length.

In [6], we have derived a simplified expression for the energy of several vortex lines in a rotating trap from the usual Gross Pitaevskii energy describing the steady state of the condensate

$$\mathcal{E}_{3D}(\phi) = \int \frac{\hbar^2}{2m}|\nabla\phi|^2 + \hbar\tilde{\Omega}\cdot(i\phi,\nabla\phi\times\mathbf{x}) + \frac{m}{2}\sum_\alpha \omega_\alpha^2 r_\alpha^2|\phi|^2 + \frac{N}{2}g_{3D}|\phi|^4. \qquad (1)$$

We let $d = (\hbar/m\omega_y)^{1/2}$ be the characteristic length, $\omega_x = \alpha\omega_y$, $\omega_z = \beta\omega_y$. We define a small non-dimensional parameter ε which characterizes the fact that we are in the Thomas Fermi regime by

$$\varepsilon^2\sqrt{\varepsilon} = \frac{d}{4\pi Na},$$

where N is the number of particles and a the scattering length. In the ENS experiment [1, 2], $\varepsilon = 1.74\ 10^{-2}$ while in the MIT experiment [4], $\varepsilon = 3.52\ 10^{-3}$. We rescale distances by $d/\sqrt{\varepsilon}$ and the chemical potential μ_0 so that the new chemical potential ρ_0 is given by

$$\rho_0 = 2\varepsilon\frac{\mu_0}{\hbar\omega_y}. \qquad (2)$$

In these units, we have $\rho_0 = 0.42$ and $\rho_0 = 0.46$ respectively for the ENS and MIT experiments. We let

$$\rho(\mathbf{r}) = \rho_0 - (\alpha^2x^2 + y^2 + \beta^2z^2) \qquad (3)$$

be the Thomas Fermi limit of the wave function in rescaled units. Then, we have obtained in [6] a simplified expression for the energy of a vortex line γ, which is

$$\varepsilon\hbar\omega_y\pi|\log\varepsilon|\ E[\gamma]$$

with

$$E[\gamma] = \int_\gamma \rho\ dl - \Omega\int_\gamma \rho^2\ dz, \qquad (4)$$

where Ω is related to the experimental rotational velocity $\tilde{\Omega}$ by

$$\Omega = \frac{\tilde{\Omega}}{\omega_y}\frac{1}{(1+\alpha^2)\varepsilon|\log\varepsilon|}. \qquad (5)$$

The energy $E[\gamma]$ reflects the competition between the vortex energy due to its length (1st term) and the rotation term. Note that the rotation term is an oriented integral (dz not dl), which actually forces the vortex to be along

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the z axis, while the other term wants to minimize the length. This is why, according to the geometry of the trap, the shape of the vortex varies.

This energy is similar to the one obtained in [8] in the study of the dynamics of the vortex line. Note that the energy that we actually derive in [6] is slightly more involved than (4). In the regime of the experiments, it is reasonable to restrict to this expression (4), taking into account the fact that the vortex core is sufficiently small (it is of size ε in our units) and neglecting the interaction of the curve with itself. We are interested only in the presence of the first vortex: when there are several vortices, the energy has an extra term due to the repulsion between the lines.

In this scaling, the energy of the vortex free solution is zero. Thus, a vortex line is energetically favorable when Ω, β are such that $\inf_\gamma E[\gamma] < 0$. The aim of this paper is to study the shape of the vortex lines γ minimizing $E[\gamma]$. We define the domain $\mathcal{D} = \{\rho > 0\}$. This is the domain where the condensate lies. All the analysis will be made in $\mathcal{D}$. In what follows, we assume that we are at a velocity Ω such that there is a vortex line and we want to find conditions on Ω and the elongation β for the line to be stable and either straight or bent.

First of all, it has been observed numerically [5] that the vortex line lies in the plane closest to the axis of rotation and we can provide a rigorous justification:

Theorem 1 *If $\alpha \leq 1$, then the energy is minimized when the vortex line lies in the (y, z) plane, that is the plane closest to the axis.*

Indeed, if we have a curve γ parametrized as $\gamma(t) = (x(t), y(t), z(t))$, then we can define the new curve $\tilde{\gamma}(t) = (0, \tilde{y}(t), \tilde{z}(t))$ by $\tilde{z}(t) = z(t)$ and $\tilde{y}(t) = \sqrt{\alpha^2 x^2 + y^2}$. Then $\rho(\gamma(t)) = \rho(\tilde{\gamma}(t))$. Since $\alpha < 1$, $\dot{\tilde{y}}^2 \leq \dot{x}^2 + \dot{y}^2$, hence $\rho(\tilde{\gamma})|\dot{\tilde{\gamma}}| - \Omega\rho(\tilde{\gamma})\dot{\tilde{z}} \leq \rho(\gamma)|\dot{\gamma}| - \Omega\rho(\gamma)\dot{z}$. It follows that the energy of the new curve $E[\tilde{\gamma}]$ is less or equal than $E[\gamma]$. If $\alpha = 1$, that is the cross section is a disc, then our arguments imply that the vortex line is planar, but of course all transversal planes are equivalent.

From now on, we will assume that the curve lies in the plane (y, z), so that ρ, given by (3), only depends on y and z. Recall from the expression of E, (4), that for $E[\gamma]$ to be negative, we need $\rho - \Omega\rho^2$ to be negative somewhere, that is $\Omega\rho > 1$. For fixed Ω, we define the regions

$$\mathcal{D}_g := \{(y, z) : \Omega\rho(y, z) > 1\}, \qquad \mathcal{D}_b := \mathcal{D} \setminus \mathcal{D}_g.$$

We will refer to these sets as "the good region" $\mathcal{D}_g$ and "the bad region" $\mathcal{D}_b$ respectively. In the bad region, the energy of a vortex per unit arc length is necessarily positive, since $\rho - \Omega\rho^2 > 0$, whereas in the good region, for appropriately oriented vortices it can be negative since $\rho - \Omega\rho^2 < 0$. One can see easily that for γ to have a negative energy, part of the vortex line has to lie in the good region, that is close to the center of the cloud. Note that for $\mathcal{D}_g$ to be non empty, we need at least $\Omega\rho_0 > 1$. In the region $\mathcal{D}_g$, we will see that the vortex is close to the axis for all β. On the other hand, in the region $\mathcal{D}_b$, the vortex goes to the boundary along the quickest path: if β is small, perpendicularly to the boundary, which gives rise to a bent vortex and if $\beta > 1$, the vortex stays along the axis of rotation.

The organization of the paper is the following: first we study the local stability of the straight vortex: if Ω is large, then the straight vortex is a local minimizer. That is when Ω gets large, the vortices tend to be straight, while if β is small then the straight vortex loses local stability and the first vortex to be nucleated is bent. Then, we study the minimization of $E[\gamma]$ in $\mathcal{D}_g$ and $\mathcal{D}_b$ according to the value of β. We finally derive that a minimizer of the energy has a minimal length.

II. STABILITY AND INSTABILITY OF THE STRAIGHT VORTEX

In this section, we study the stability of the straight vortex. We parametrize the straight vortex as $\gamma_s(z) = (0, z)$ for $-z_{\max} < z < z_{\max}$, with $z_{\max} = \sqrt{\rho_0}/\beta$. One can compute $E[\gamma_s]$ and derive that it is 0 for $\Omega\rho_0 = 5/4$. We have two aims: first show that for β small, when the straight vortex has 0 energy or small negative energy, that is for $\Omega\rho_0$ close to $5/4$, then it is unstable. Then, we want to prove on the contrary that if β is fixed and Ω is sufficiently big, the straight vortex is stable.

We consider perturbations of the straight vortex of the form $\gamma_\delta(z) = (\delta v(z), z + \delta^2 w(z)) + O(\delta^3)$ for $|z| < z_{\max}$. We require that w be chosen so that $\rho(\gamma_\delta(\pm z_{\max})) = 0$, thereby respecting the condition that the vortex line terminate at the boundary of the cloud.

Writing a Taylor series expansion for E, one finds that

$$E[\gamma_\delta] = E[\gamma_s] + \frac{\delta^2}{2}(v, E''[\gamma_s]v) + O(\delta^3)$$

where

$$(v, E''[\gamma_s]v) = \int_{-z_{\max}}^{z_{\max}} 2(2\Omega\rho - 1)v^2 + \rho v'^2 dz.$$

Here and in the rest of this section, $\rho = \rho(0, z) = \rho_0 - \beta^2 z^2$. To get this it is necessary to integrate by parts and use the fact that the straight vortex solves the Euler-Lagrange equations for E. In particular this eliminates all terms involving w. No boundary terms arise from integration by parts because $\rho(\gamma_\delta) = 0$ at the endpoints.

We say that the straight vortex is stable if $(v, E''[\gamma_s]v) > 0$ for all v, and unstable if $(v, E''[\gamma_s]v) < 0$ for some v.

Theorem 2 *The straight vortex is stable if*

$$\Omega\rho_0 > \frac{3}{4} + \frac{1}{4\beta^2}.$$

The straight vortex is unstable if $\beta < 1/\sqrt{3}$ and

$$\Omega\rho_0 < \frac{1}{6} + \frac{1}{6\beta^2}. \tag{6}$$

Note that the 2 values are consistent in the sense that they both scale like $1/\beta^2$ when β is small. For Ω large, one expects several vortices in the condensate, but the fact that a straight vortex is stable gives an indication that for Ω large, each vortex should be nearly straight, which is consistent with the observations [3].

Remark 1 *It is interesting to see what happens in Theorem 2 when $\Omega\rho_0 = 5/4$, that is when the straight vortex has zero energy. The first inequality yields that if $\beta > 1/\sqrt{2}$, then the straight vortex is stable for all Ω such that $\Omega\rho_0 > 5/4$, that is when $E[\gamma_s] < 0$. If $\beta > 1$, we will see that γ_s is not just stable but in fact minimizes E. The second inequality implies that, if $\beta < \sqrt{2/13} \approx .39$ then the straight vortex is unstable at the velocity $\Omega\rho_0 = 5/4$ at which $E[\gamma_s] = 0$. As a result, for these values of β, the first vortex to nucleate as Ω increases is a bent vortex. Note that it has been observed in [8] that for $\beta < 1/2$, the ground state of the system corresponds to a curved line.*

All this indicates that by varying the elongation of the condensate, one may hope to go from a situation where the first vortex is bent, to a situation where it is straight.

To prove the instability of the straight vortex, we will find explicit perturbations v for which $(v, E''[\gamma_s]v) < 0$. These also indicate the shape of good test functions.

We define a perturbation v (depending on a parameter θ, which for now we regard as fixed) by

$$v(z) = \begin{cases} 0 & \text{if } z \le \theta z_{\max} \\ \left(\frac{z}{z_{\max}} - \theta\right)(1-\theta)^{-1} & \text{if } z \ge \theta z_{\max}. \end{cases}$$

Here v is normalized so that $v(z_{\max}) = 1$. For this choice of v, a lengthy but straightforward calculation shows that

$$\begin{aligned} (v, E''[\gamma_s]v) \quad &= \frac{2\Omega\rho_0^{3/2}}{30\beta}[(1-\theta)^2(\theta+4) \\ &\quad -\frac{5}{\Omega\rho_0}(1-\theta) - \beta^2(1+\frac{\theta}{2})] \\ &=: \frac{2\Omega\rho_0^{3/2}}{30\beta}\Delta(\theta). \end{aligned}$$

It follows that the straight vortex is unstable if

$$(1-\theta)^2(\theta+4) < \frac{5}{\Omega\rho_0}\left((1-\theta) - \beta^2(1+\frac{\theta}{2})\right) \tag{7}$$

for some $\theta \in [0, 1)$. It is helpful to write θ as $\theta = 1 - \eta\beta^2$ for some $\eta > 0$ to be determined. Then (7) can be written in terms of η, as

$$\Omega\rho_0 < 5\left(\frac{1 + (\beta^2/2) - (3/2\eta)}{\eta\beta^2(5 - \eta\beta^2)}\right).$$

This is satisfied if

$$\Omega\rho_0 < \frac{1 + (\beta^2/2) - (3/2\eta)}{\eta\beta^2} = \frac{1}{2\eta} + \frac{1}{\eta\beta^2}(1 - \frac{3}{2\eta}).$$

The extremum is achieved for η close to 3, so we can take $\eta = 3$ to find that (6) is a sufficient condition for instability. Because $\theta = 1 - \eta\beta^2 \ge 0$, this conclusion only holds if $\beta \le 1/\sqrt{3}$. For larger values of β, one can make different choices of θ to find thresholds for instability.

To derive the sufficient condition for stability, note that for every z,

$$\frac{3\rho}{2\rho_0} - \frac{(z\rho)'}{2\rho_0} = 1.$$

Multiplying v^2 by the expression on the left and integrating by parts, we obtain

$$\int_{-z_{\max}}^{z_{\max}} v^2 dz = \int_{-z_{\max}}^{z_{\max}} \rho\left[\frac{3v^2}{2\rho_0} + \frac{z}{\rho_0}vv'\right] dz.$$

Since $|z|/\rho_0 \le z_{\max}/\rho_0 = 1/\beta\sqrt{\rho_0}$ for $|z| < z_{\max}$,

$$\int_{-z_{\max}}^{z_{\max}} v^2 dz \le \int_{-z_{\max}}^{z_{\max}} \rho\left[\frac{3}{2\rho_0}v^2 + \frac{1}{\beta\sqrt{\rho}_0}|v|\,|v'|\right] dz.$$

Now we use the inequality $ab \le a^2/2 + b^2/2$, to deduce

$$\int_{-z_{\max}}^{z_{\max}} v^2 dz \le \int_{-z_{\max}}^{z_{\max}} \rho\left[(\frac{3}{2\rho_0} + \frac{1}{2\rho_0\beta^2})v^2 + \frac{1}{2}(v')^2\right] dz.$$

In particular, if

$$\Omega\rho_0 > \frac{3}{4} + \frac{1}{4\beta^2}$$

then this implies that $(v, E''[\gamma_s]v) > 0$ for all v. This completes the proof of Theorem 2.

We would like to derive a more precise estimate of the critical velocity for which a bent vortex minimizes the energy $E[\gamma]$. We have seen that for $E[\gamma]$ to be negative, we need at least $\Omega\rho_0 > 1$ so that the good region $\mathcal{D}_g$ is nonempty. Note that $\Omega\rho_0 = 1$ is exactly the 2D critical velocity at the plane $z = 0$ for the existence of a vortex. But a bent vortex cannot be a minimizer of $E[\gamma]$ exactly at $\Omega\rho_0 = 1$, since the good region $\mathcal{D}_g$ has to have some critical size so that the vortex energy in the good region provides a sufficient contribution to compensate the positive part due to the length in the bad region. On the other hand, for $\Omega\rho_0 = 5/4$, the straight vortex has 0 energy. Thus, the critical velocity to obtain a bent vortex is $1 < \Omega_c\rho_0 < 5/4$. We want to obtain a sharper estimate by using appropriate test functions. To find good test functions, note that

$$\Delta'(\theta) = [3\theta^2 + 4\theta - (7 - \frac{5}{2\Omega\rho_0}(2 + \beta^2)]$$

and so Δ has a local maximum at

$$\theta_* = -\frac{2}{3} + \sqrt{\frac{25}{9} - \frac{5}{6\Omega\rho_0}(1-\beta^2)}$$

which lies in the interval $(0,1)$ for the parameter range that we care about.

Note that θ_* is an increasing function of Ω, which is consistent with numerical calculations showing that for larger values of Ω, the minimizing path stays close to the z axis over a longer interval. For $\theta = \theta_*$, we compute the energy of the path which is straight between $z=-\theta$ and θ and goes to the boundary along a straight line. The optimal end point on the boundary is at $z = \theta+\beta$ for β small. For this special test function γ, we can compute $E[\gamma]$ to find that it is less than

$$\frac{\Omega\rho_0}{8}\left(\frac{53}{4} - 36\beta + 21\beta^2 - 4\beta^3\right) - \frac{25}{8} + 3\beta - \beta^2 + 10 - 8\Omega\rho_0$$

Thus, for β small, we find an upper bound for the critical velocity which yields a negative energy for such a test function:

$$\Omega\rho_0 = \frac{(220+96\beta)}{(203+76\beta)}.$$

In the condition of the ENS experiment, this yields $\Omega\rho_0 < 1.08$, that is in the original variable (see (5)), $\tilde{\Omega}/\omega_y < 0.385$, which is very close to the value found numerically 0.38 [7].

As a conclusion, we have shown that there is a critical value of Ω called Ω_c with $\Omega_c\rho_0 \approx 1.08$, such that a bent vortex has negative energy and less energy than a straight vortex.

III. SHAPE OF THE VORTEX ACCORDING TO β

In this section we prove that when the condensate cloud has a pancake shape, then the straight vortex is always minimizing among vortices with negative energy.

Recall that $\mathcal{D} = \{\rho > 0\}$ and we write $\gamma(t) = (y(t), z(t))$ to denote a generic vortex line represented by a continuous Lipschitz function from $I = [0,1]$ into $\overline{\mathcal{D}}$ such that $\gamma(0), \gamma(1) \in \partial\mathcal{D}$.

For such a curve γ, let $I_{\gamma,g} := \{t \in I : \gamma(t) \in \mathcal{D}_g\}$ and $I_{\gamma,b} = I \setminus I_{\gamma,g}$. And let γ_g be the restriction of $\gamma(\cdot)$ to $I_{\gamma,g}$, and similarly γ_b.

The definition of $I_{\gamma,b}$ implies that $\rho(\gamma(t)) - \Omega\rho^2(\gamma(t)) > 0$ for $t \in I_{\gamma,b}$, and as a consequence

$$\rho(\gamma(t))|\dot\gamma(t)| - \Omega\rho^2(\gamma(t))\dot z \geq |\dot\gamma(t)|\left(\rho(\gamma(t)) - \Omega\rho^2(\gamma(t))\right)$$

which is positive in $I_{\gamma,b}$. Thus if γ is such that $I_{\gamma,g}$ is empty, then clearly $E[\gamma] > 0$ and it is energetically favorable not to have a vortex. This is the case in particular for $\Omega\rho_0 < 1$ since then $\mathcal{D}_g$ is empty. We may thus restrict our attention to the case $I_{\gamma,b}$ nonempty.

Proposition 1 *Let $M_g = \inf\{E[\gamma_g]\}$, where γ_g is the restriction of $\gamma(\cdot)$ to $I_{\gamma,g}$, then for all β and all Ω, the infimum is attained by the straight vortex.*

Proposition 2 *Let $M_b = \inf\{E[\gamma_b]\}$, , where γ_b is the restriction of $\gamma(\cdot)$ to $I_{\gamma,b}$, then for all $\beta \geq 1$, the infimum is attained by the straight vortex.*

Note that in the bad region, Proposition 2 only holds for $\beta > 1$. If $\beta < 1$, the situation is somewhat more complicated: $\int_{\gamma_b} \rho dl$ is minimized by a path that joins $\mathcal{D}_g$ to $\partial\mathcal{D}$ along the y axis, whereas $-\int_{\gamma_b}\rho^2 dz$ is minimized by the straight vortex running along the z-axis. The minimizer of the full energy reflects the competition between these two terms, and hence is bent.

We always have

$$E[\gamma] = E[\gamma_g] + E[\gamma_b] \geq M_g + M_b$$

In particular, as a corollary of the above propositions we deduce

Theorem 3 *For $\beta \geq 1$*

$$E[\gamma] \geq \inf(0, E[\gamma_s]) \tag{8}$$

where γ_s is the straight vortex along the z axis. If $E[\gamma_s] < 0$, the equality in (8) can happen only if γ is the straight vortex.

To prove Proposition 1, first note that

$$\int_{\gamma_g} \rho dl - \Omega\rho^2 dz \geq \int_{\gamma_g} \rho|dz| - \Omega\rho^2 dz \geq \int_{\gamma_g}(\rho - \Omega\rho^2)dz.$$

Since we have assumed that γ does not self-intersect, we can identify γ with the (oriented) boundary of an open set $V \subset \mathcal{D}$. Then γ_g can be identified with $\mathcal{D}_g \cap \partial V = \partial(\mathcal{D}_g \cap V) \setminus (\partial\mathcal{D}_g \cap \bar V)$. Since $\rho - \Omega\rho^2 = 0$ precisely on $\partial\mathcal{D}_g$, this implies that

$$\int_{\gamma_g}(\rho - \Omega\rho^2)dz \;=\; \int_{\partial(\mathcal{D}_g\cap V)}(\rho-\Omega\rho^2)dz.$$

And by Stokes' Theorem,

$$\int_{\partial(\mathcal{D}_g\cap V)}(\rho-\Omega\rho^2)dz \;=\; \int_{\mathcal{D}_g\cap V}(1-2\Omega\rho)\rho_y dydz.$$

The definition of $\mathcal{D}_g$ implies that $1 - 2\Omega\rho < 0$, and so this integral is clearly minimized if $\mathcal{D}_g \cap V$ is just the subset of $\mathcal{D}_g$ where $\rho_y > 0$, so that

$$\int_{\partial(\mathcal{D}_g\cap V)}(\rho-\Omega\rho^2)dz \;\geq\; \int_{\{(y,z)\in\mathcal{D}_g : y<0\}}(1-2\Omega\rho)\rho_y dydz. \tag{9}$$

Again using Stokes Theorem and the fact that $\rho - \Omega\rho^2$ vanishes on $\partial\mathcal{D}_g$, we find that this is equal to

$$\int_{-z_*}^{z_*}\Big(\rho(0,z) - \Omega\rho^2(0,z)\Big)dz,$$

where $(0, \pm z_*)$ are the points where the z-axis intersects $\partial\mathcal{D}_g$. Combining these inequalities, we find that

$$\int_{\gamma_g} \rho\ dl - \Omega\rho^2\ dz \geq \int_{-z_*}^{z_*} \Big(\rho(0,z) - \Omega\rho^2(0,z)\Big)\ dz. \quad (10)$$

It is easy to see that equality holds in (9), and hence in (10), exactly when γ is the straight vortex, and so we have proved Proposition 1.

To prove Proposition 2, fix γ such that $I_{\gamma,g}$ is nonempty. The beginning and end of γ must lie in the bad region, and γ intersects the good region, so $I_{\gamma,b}$ must consist of at least two components. Let (a_1, b_1) denote the first such component and (a_2, b_2) denote the last, and write γ_1 and γ_2 to denote the corresponding portions of γ_b, so that γ_1 is parametrized as $\gamma_1 = (y, z) : (a_1, b_1) \to \mathcal{D}_b$, with $\gamma_1(a_1) \in \partial\mathcal{D}$ and $\gamma_1(b_1) \in \partial\mathcal{D}_g$. We need to show that γ_1 and γ_2 both have more energy than the corresponding parts of the straight vortex. We will consider only γ_1, as the argument for γ_2 is exactly the same.

Define $\gamma_s = (0, \zeta)$ to be a parametrization of the part of the straight vortex joining $(0, -z_{\max})$ to $(0, -z_*)$, where $z_{\max} = \sqrt{\rho_0}/\beta$:

$$\tilde{\zeta}(t) = -\frac{1}{\beta}(y(t)^2 + \beta^2 z(t)^2)^{1/2},\ \zeta(t) = \max_{a \leq s \leq t} \tilde{\zeta}(s).$$

Recall that we have $\gamma_1 = (y(t), z(t))$. The definition is arranged so that $t \mapsto \zeta(t)$ is nondecreasing and $|\dot{\gamma}_s| = \dot{\zeta}$. To prove the proposition, it thus suffices to show that

$$\rho(\gamma_1)|\dot{\gamma}_1| - \Omega\rho^2(\gamma_1)\dot{z} \geq \rho(\gamma_s)|\dot{\gamma}_s| - \Omega\rho^2(\gamma_s)\dot{\zeta}.$$

If $\zeta(t) > \tilde{\zeta}(t)$, this is clear, because then $\dot{\zeta} = 0$, so the right-hand side vanishes while the left-hand side is nonnegative, by the defining property of the bad region $\mathcal{D}_b$.

And if $\zeta(t) = \tilde{\zeta}(t)$, then $\rho(\gamma_1(t)) = \rho(\gamma_s(t))$, and so in this case $0 \leq 1 - \Omega\rho(\gamma_1(t)) = 1 - \Omega\rho(\gamma_s(t)) \leq 1$. So we only need to show that

$$|\dot{\gamma}| - c\dot{z}\ \geq\ |\dot{\gamma}_s| - c\dot{\zeta} \quad (11)$$

for any $c \in [0, 1]$. We will apply it to $c = \Omega\rho(\gamma_s(t))$.

To do this, first note that

$$\dot{\zeta} = \dot{\tilde{\zeta}} = \frac{1}{\tilde{\zeta}}\left(\frac{y\dot{y}}{\beta^2} + z\dot{z}\right) = (\dot{y}, \dot{z}) \cdot \left(\frac{1}{\tilde{\zeta}}(\frac{y}{\beta^2}, z)\right).$$

So

$$|\dot{\zeta}|\ \leq\ |\dot{\gamma}|\left(\frac{1}{\tilde{\zeta}^2}(\frac{y^2}{\beta^4} + z^2)\right)^{1/2} = |\dot{\gamma}|\left(\frac{\beta^{-4}y^2 + z^2}{\beta^{-2}y^2 + z^2}\right)^{1/2}.$$

Since $\beta > 1$, we conclude that $|\dot{\zeta}| \leq |\dot{\gamma}_1|$. Also, it is clear that $|\dot{z}| \leq |\dot{\gamma}_1|$. So if $0 \leq \alpha \leq 1$, then

$$|\dot{\gamma}_1| - c\dot{z} \geq |\dot{\gamma}_1|(1 - c) \geq \dot{\zeta}(1 - c) = |\dot{\gamma}_s| - c\dot{\zeta},$$

which proves (11), and hence Proposition 2.

IV. MINIMAL LENGTH

In the case $\beta < 1$, that is when the vortex line is bent, we will prove that the vortex has a minimum length. This is related to the fact that the vortex has to go to the center of the cloud and spend some time in the good region.

For an open set $U \subset \mathcal{D}$ with Lipschitz boundary, we endow ∂U with an orientation in the standard way, so that Stokes' theorem holds.

We will prove the following isoperimetric-type inequality:

Theorem 4 *For every* $0 < \beta \leq 1$

$$\left|\int_{\partial U} \rho^2 dz\right|\ \leq\ (2\sqrt{\rho_0})^{1/2}\left(\int_{\partial U} \rho dl\right)^{3/2} \quad (12)$$

for every connected open subset $U \subset \mathcal{D}$,

Remark 2 *The exponent* $3/2$ *is the best possible. An inequality similar to (12) is valid for* $\beta > 1$*, but the proof needs to be modified a bit. For the straight radial vortex,*

$$\int_{\partial U} \rho^2 dz = \frac{16}{15}\frac{(\rho_0)^{5/2}}{\beta} \quad \text{and} \quad \int_{\partial U} \rho dl = \frac{4}{3}\frac{(\rho_0)^{3/2}}{\beta},$$

and so

$$\left(\int_{\partial U} \rho^2 dz\right)\left(\int_{\partial U} \rho dl\right)^{-3/2} \approx 0.52\ \beta^{1/2}(\rho_0)^{1/4}.$$

This shows that the constant $(2\sqrt{\rho_0})^{1/2}$ *in (12) is fairly close to sharp for* $\frac{1}{4} \leq \beta < 1$ *say.*

1. We use Stokes' Theorem to calculate

$$\int_{\partial U} \rho^2 dz = 2\int_U \rho\rho_y dydz \leq 2\int_{U^-} \rho\rho_y dydz$$

where $U^- = \{(y, z) \in U\ :\ y < 0\}$, since $\rho\rho_y \leq 0$ for $(y, z) \in \mathcal{D}$ such that $y \geq 0$.

So the coarea formula implies that

$$\int_{\partial U} \rho^2 dz \leq 2\int_{U^-} \rho\frac{|\rho_y|}{|\nabla\rho|}|\nabla\rho| dydz$$
$$= 2\int_{\rho_*}^{\rho^*} s\left(\int_{\{(y,z)\in U^-\ :\ \rho(y,z)=s\}} \frac{|\rho_y|}{|\nabla\rho|}\ dl\right) ds$$

where $\rho_* = \inf\{\rho(y, z)\ :\ (y, z) \in U\}$, and $\rho^* = \sup\{\rho(y, z)\ :\ (y, z) \in U\}$. Thus

$$\left|\int_{\partial U} \rho^2 dz\right|\ \leq\ |\rho^* - \rho_*|$$
$$\sup_s\left(s\int_{\{(y,z)\in U\ :\ \rho(y,z)=s\}} \frac{\rho_y}{|\nabla\rho|}\ dl\right).$$

Thus to prove the theorem it suffices to establish the following two claims:

$$s\int_{\{(y,z)\in U\ :\ \rho(y,z)=s\}}\frac{\rho_y}{|\nabla\rho|}dl\ \le\ \int_{\partial U}\rho dl \tag{13}$$

for every s, and

$$|\rho^*-\rho_*|\le(2\sqrt{\rho_0})^{1/2}\left(\int_{\partial U}\rho dl\right)^{1/2}. \tag{14}$$

2. We first prove (13). Fix some $s\in(\rho_*,\rho^*)$ and write Γ_s to denote $\{(y,z)\in U^-\ :\ \rho(y,z)=s\}$. Also, let $\tilde\Gamma_s$ denote $\partial U\cap\{\rho\ge s\}$.

First assume for simplicity that Γ_s is connected, so that it consists of the short arc of the ellipse $\{\rho=s\}$ joining two points, say $p_0=(y_0,z_0)$ and $p_1=(y_1,z_1)$ with $z_0<z_1$. We can represent Γ_s as the image of the mapping

$$z\mapsto(y(z),z)=(-(s-\beta^2z^2)^{1/2},z),\qquad z_0<z<z_1.$$

Differentiating the identity $\rho(y(z),z)=s$ we find that $\rho_y y'(z)+\rho_z=0$. Thus

$$\left|\frac{d}{dz}(y(z),z)\right|=\left(1+y'(z)^2\right)^{1/2}=\left(\frac{(\rho_y^2+\rho_z^2)}{\rho_y^2}\right)^{1/2}=\frac{|\nabla\rho|}{|\rho_y|}.$$

It follows that

$$s\int_{\{(y,z)\in U\ :\ \rho(y,z)=s\}}\frac{\rho_y}{|\nabla\rho|}dl\ =\ s\int_{z_0}^{z_1}dz.$$

On the other hand, the one-dimensional measure of $\tilde\Gamma_s$ is certainly greater than $|p_1-p_0|\ge z_1-z_0$, and $\rho\ge s$ on $\tilde\Gamma_s$, and so

$$\int_{\tilde\Gamma_s}\rho(z,y)dl\ \ge\ s\ l(\tilde\Gamma_s)\ \ge\ s(z_2-z_1).$$

This proves (13) if Γ_s is connected. If not, one can apply the same argument on each connected component of Γ_s.

3. Next we prove (14). Let q_* and q^* be points in ∂U such that $\rho(q_*)=\rho_*$, $\rho(q^*)=\rho^*$. Since we have assumed that U is connected, ∂U contains a path joining q_* to q^*. In fact it contains two such paths. If we write $\mathcal{P}$ to denote the set of all Lipschitz paths in $\mathcal{D}$ joining the level set $\{\rho=\rho_*\}$ and the level set $\{\rho=\rho^*\}$, it follows that

$$\int_{\partial U}\rho\ dl\ \ge 2\ \inf_{\gamma\in\mathcal{P}}\int_\gamma\rho\ dl.$$

Arguments in the proof of Proposition 2 show that for $\beta\le1$, $\inf_{\gamma\in\mathcal{P}}\int_\gamma\rho dl$ is attained by a path that goes in a straight line along the y axis. Thus

$$\inf_{\gamma\in\mathcal{P}}\int_\gamma\rho\ dl=\int_{y^*}^{y_*}(\rho_0-y^2)\ dy,$$

where $y_*=\sqrt{\rho_0-\rho_*}$, $y^*=\sqrt{\rho_0-\rho^*}$. And since $y_*,y^*\le\sqrt{\rho}_0$,

$$\begin{aligned}\int_{y^*}^{y_*}(\rho_0-y^2)\ dy\quad &\ge\frac{1}{2\sqrt{\rho_0}}\int_{y^*}^{y_*}(\rho_0-y^2)2y\ dy\\ &=\frac{1}{2\sqrt{\rho_0}}\int_{\rho_*}^{\rho^*}\rho d\rho\\ &=\frac{1}{4\sqrt{\rho_0}}\left((\rho^*)^2-(\rho_*)^2\right).\end{aligned}$$

Since $b^2-a^2\ge(b-a)^2$ when $0<a<b$, we deduce that (14) holds. This concludes the proof of the theorem.

A short calculation starting from (12) shows that if $E[\gamma]<0$ then

$$\int_\gamma\rho dl>\frac{1}{(2\Omega^2\sqrt{\rho_0})}. \tag{15}$$

We expect that even for a configuration with multiple vortices, adding a new vortex would require to overcome an energy barrier which implies that the length of the new vortex has a lower bound of the type (15).

V. CONCLUSION

We have studied the shape of the first vortex line to be nucleated in a harmonic anisotropic rotating potential, according to Ω and the elongation of the cloud β. We investigate the stability of the straight vortex and get that when Ω is large, the straight vortex is a local minimum of the energy. We prove that when a vortex is nucleated, it is close to the axis of rotation where the condensate density is high, and that near the boundary, where the density is low, the shape of the vortex depends on whether the cloud has a cigar or pancake shape. This shape reflects the competition in the energy between the rotation and the inhomogeneity of the trap, which makes the geometry of the experiment very important. In the case $\beta>1$ (pancake), the vortex stays straight along the z axis while in the case β small (cigar), the vortex is bending. In the case β small, this allows us to define an energy barrier for the nucleation of vortices and to prove that when a vortex line is nucleated, it has to have a minimal length.

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